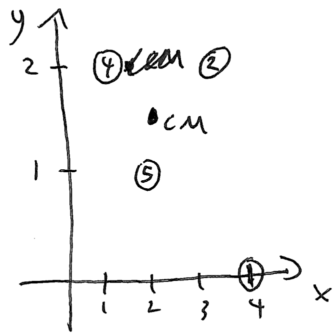
MA114 Summer 2018
Worksheet 21 – Centers of Mass – 7/19/18

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates (1, 2), (3, 2), (2, 1), and (4, 0).



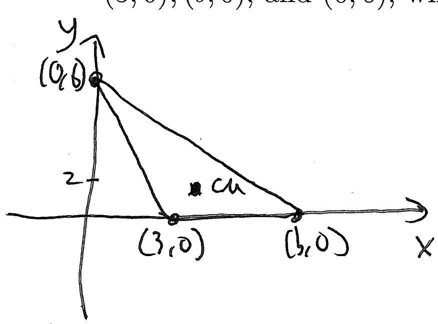
$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{4\cdot 1 + 2\cdot 3 + 5\cdot 2 + 1\cdot 4}{4+2+5+1} = \frac{24}{12} = 2$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{4\cdot 2 + 2\cdot 2 + 5\cdot 1 + 1\cdot 0}{4+2+5+1} = \frac{17}{12}$$

$$\boxed{\left(2, \frac{17}{12}\right)}$$

2. Point masses of equal size are placed at the vertices of the triangle with coordinates (3, 0), (b, 0), and (0, 6), where $b > 3$. Find the center of mass.

Let m be the mass of each point mass



$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{3\cdot m + b\cdot m + 0\cdot m}{3m}$$

$$= 1 + \frac{b}{3}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{6\cdot m + 0\cdot m + 0\cdot m}{3m} = 2$$

$$\boxed{\left(1 + \frac{b}{3}, 2\right)}$$

3. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \le x \le 1$.

$$M = \int_0^1 1 - x^2\,dx = x - \frac{1}{3}x^3\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$M_y = \int_0^1 x(1-x^2)\,dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$M_x = \int_0^1 \frac{1}{2}(1-x^2)^2\,dx = \frac{1}{2}\int_0^1 1 - 2x^2 + x^4\,dx = \frac{1}{2}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{1}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{4}{15}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{1}{4}}{\frac{2}{3}}, \frac{\frac{4}{15}}{\frac{2}{3}}\right) = \left(\frac{3}{8}, \frac{12}{30}\right) = \boxed{\left(\frac{3}{8}, \frac{2}{5}\right)}$$

4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.

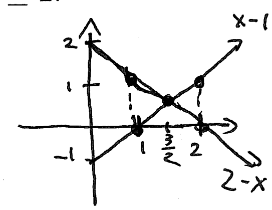
$$M = \int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}\cdot 7 = \frac{14}{3}$$

$$M_y = \int_1^4 x\sqrt{x}\,dx = \frac{2}{5}x^{5/2}\Big|_1^4 = \frac{2}{5}(32-1) = \frac{62}{5}$$

$$M_x = \int_1^4 \frac{(\sqrt{x})^2}{2}\,dx = \frac{1}{4}x^2\Big|_1^4 = \frac{1}{4}(16-1) = \frac{15}{4}$$

$$(\bar{x},\bar{y}) = \left(\frac{62/5}{14/3}, \frac{15/4}{14/3}\right) = \left(\frac{\cancel{62}^{\,31}}{5}\cdot\frac{3}{\cancel{14}_{\,7}}, \frac{15}{4}\cdot\frac{3}{14}\right) = \left(\frac{93}{35}, \frac{45}{56}\right)$$

5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $1 \leq x \leq 2$.



$\cancel{M = \int_1^2}$ By symmetry, $(\bar{x},\bar{y}) = \left(\frac{3}{2},\frac{1}{2}\right)$.

Otherwise, by symmetry $M = 2\cdot\int_{3/2}^2 (x-1)-(2-x)\,dx = 2\int_{3/2}^2 2x-3\,dx$
$= 2\left(x^2-3x\right)\Big|_{3/2}^2$
$= 0.5$

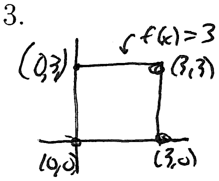
$$M_y = \int_1^{3/2} x\big((2-x)-(x-1)\big)\,dx + \int_{3/2}^2 x\big((x-1)-(2-x)\big)\,dx$$
$$= \frac{7}{24} + \frac{11}{24} = \frac{18}{24} = \frac{3}{4}$$

Then $\bar{x} = \frac{3/4}{1/2} = \frac{3}{2}$

$$M_x = \frac{1}{2}\int_1^{3/2} (2-x)^2 - (x-1)^2\,dx + \frac{1}{2}\int_{3/2}^2 (x-1)^2 - (2-x)^2\,dx$$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$\bar{y} = \frac{1/4}{1/2} = \frac{1}{2}$.

6. Bonus Fun Problem: Find the mass of a square plate with vertices at $(0, 0)$, $(3, 0)$, $(0, 3)$, and $(3, 3)$ with changing density function $\rho(x) = x + 5$ for $0 \leq x \leq 3$.



From class, $M = \int_a^b \rho(x)\cdot f(x)\,dx$
$$= \int_0^3 (x+5)\cdot 3\,dx$$
$$= 3\left(\frac{x^2}{2}+5x\right)\Big|_0^3$$
$$= 3\left(\frac{9}{2}+15\right)$$
$$= \boxed{\frac{117}{2}}$$